SOLUTION OF TRANSIENT MARKOVIAN STATE PROBABILITIES AND RATE OF APPROACH TO EQUILIBRIUM STATES

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Abstract. The differential equations for transient state probabilities for Markovian processes are examined to derive the rate of convergence of transient states to equilibrium states. There is an acute need to solve the balance equations for large states, particularly for handling computer performance modeling with a network of queues that do not satisfy product form solutions or cannot be cast into the forms convenient for mean value analysis.

The rate of convergence to equilibrium states is derived for irreducible aperiodic homogeneous Markov chains on the basis of a geometrical interpretation. A numerical integration method with dynamic step-size adjustments is applied and compared against the power method of Wallace and Rosenberg.

Keywords. Queuing theory; computer evaluation; Markov processes; systems analysis; numerical methods.

INTRODUCTION

In recent years much progress has been made in computer performance modeling with networks of queues representing computer system components under various statistical and queuing assumptions. Modeling approaches generally assume steady states and Markovian processes.

The equilibrium conditions for finite state systems can often be expressed in terms of the relevant state transition rates and state probability vectors. However, it is sometimes an arduous task to write transition rates explicitly in terms of basic model parameters and queuing disciplines. Computer calculations for solving global balance equations for realistic systems may become very difficult in practice due to the requirements of long computation time and large storage size.

So far, analytic investigations have concentrated on a wide variety of problems endowed with special properties such as local balance and product form solutions. But many important networks of queues do not exhibit these properties. So far, only crude approximation techniques of limited applicability exist for queuing problems involving priorities and certain types of queue disciplines.

In principle, numerical techniques may be applicable to any type of queuing network. Wallace and Rosenberg (1966) applied direct numerical iterative procedures to their development of the Recursive Queue Analyzer, RQA-1, which was used to predict computer system behavior. A more recent paper by Stewart (1978) discusses further advances made and summarizes limitations of various numerical techniques applicable to computer system modeling.

The present paper examines the notion that for a Markovian process the state transitions take place in such a manner that the final equilibrium state becomes insensitive to the initial state. In contrast to the conventional methods of obtaining numerical solutions for large states (Muntz, 1978), it is shown that transient state equations are often amenable to efficient numerical integrations starting from a crude estimate of initial states because of the Markovian nature. Quantitative results are derived to relate the desired accuracy, transition rates, and the required upper limit of integration for reaching the equilibrium state within the specified error.

PROBABILITY FLUX CONSERVATION

Typical queuing networks for modeling computer systems involve multiclass open, closed, or mixed chains, priority assignments and queue disciplines. A queuing state may be specified generally by a set of nonnegative integers. Systems that can be modeled by a continuous-time Markov process with finite discrete states are considered in this work. It is assumed that there is a one-to-one correspondence between the queuing state characterization and a set of integers, 1 to N. Any invalid states are excluded from state enumeration.

For notation, $P_i(t)$ denotes the probability that the system be found in queuing state i at time t. Further r_{ij} stands for the transition rate from state i to state j and is assumed to be explicitly independent of time. The time-dependent probability vector p satisfies the probability conservation equation:

$$\frac{dp_j}{dt} = \sum_{j \neq i} r_{ji} p_j - p_i \sum_{j \neq i} r_{ij} \qquad (1)$$

for i = 1, 2, . . . ,N where N is the total number of feasible queuing states.

Since only N-1 of the above equations are independent, the normalization condition for p is necessary for completeness. Once the probability vector is determined, the rest of the calculations for performance measures such as queue length, utilization, throughput, and response time may be carried out routinely after a mapping from the single-index state characterization back to an appropriate original queuing state specification.

SOLUTION OF PROBABILITY VECTORS

The steady state equations with the normalization condition can be cast into a matrix form

$$A\,p = e \tag{2}$$

where A is a matrix derivable from r_{ij} and e is a vector of N-1 zeros and a one. For large state spaces, it is necessary to avoid matrix inversion and devise iterative schemes by exploiting the sparseness and band structure of the transition matrix.

In this work, time-dependent equations are examined even when only the equilibrium solutions are desired.

Consider an irreducible aperiodic Markov chain with n finite time increments. The interval dt is assumed to be small enough to preclude multiple state transitions.

The transient equation can be expressed as

$$\frac{dp}{dt} = R\,p$$

where R is not explicitly dependent on time and is related to r_{ij} s through

$$R_{ij} = r_{ji} - \delta_{ij} \sum_k r_{ik} \tag{3}$$

where δ_{ij} is the Kronecker delta symbol. To simplify notation, r_{ij} is defined to be zero for j = i without losing any generality.

The transition probability matrix T is defined as the probability that the Markov process changes from the *initial state j to final state i in time dt*. Then

$$p + dp = T\,p$$

where T = 1+ R dt by definition. T satisfies the normalization relation

$$\sum_{j=1}^{N} T_{ji} = 1$$

In order to understand the time-dependent behavior, we develop a *geometrical interpretation*. For this purpose, a simplex of N vertices embedded in N-1 dimensional real space is considered. This simplex has an altitude of 1 above every base. A state probability vector p is represented by a point in the interior of the simplex. In the "barycentric coordinate system", each component of the vector p is the projection of the point on a base. The vertices correspond to N distinct Markovian states.

For example, in a 3-state problem the simplex is an equilateral triangle with an altitude of 1 in a two-dimensional plane. A Markovian state probability vector is indicated by a point in the interior, side, or vertex of the triangle. The perpendiculars to the sides are the three components of p and sum to 1. The state probability diagram is shown in Fig. 1. The three vertices of the largest triangle in Fig. 1 correspond to three Markovian states. The transition matrix T transforms any interior point of the largest triangle into a point within the next large triangle which lies within the original triangle. The vertices of the largest triangle transform to the vertices in the second largest triangle whose vertices are indicated by the three rows of T. Successive applications of the transformation T will produce successively smaller inner triangles surrounding a fixed point of the mappings.

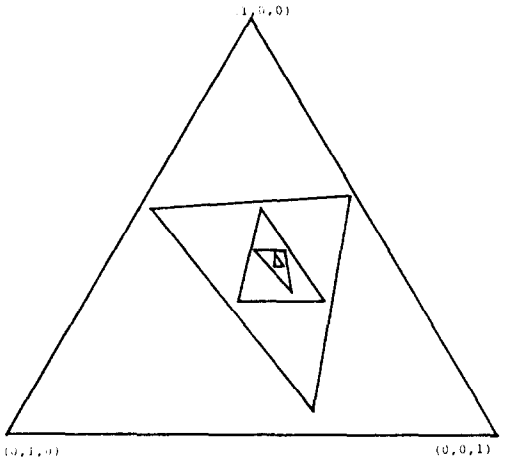

Fig. 1. Shrinkage of the Region of Uncertainty

The rate of shrinkage of the successive triangles can be derived as follows:

The area A of the triangle whose vertices are represented by (T_{11},T_{12},T_{13}), (T_{21},T_{22},T_{23}) and (T_{31},T_{32},T_{33}) may be expressed as

$$A = C \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ 1 & 1 & 1 \end{vmatrix}$$

where C is a constant which depends on the orientation of the oblique coordinate system used and the dimensionality of T. The vertical lines denote the magnitude of the determinant.

Because of the normalization relations and general properties of determinants, the last row may be replaced by T_{31},T_{32},T_{33}. Thus, A may be written as

$$A = C\,|T|$$

If T is applied n times on p, the original triangle shrinks to an inner triangle of area A(n) where

$$A(n) = C\,|T^n| = C\,|T|^n$$

The shrinkage factor for each application of T is given by

$$A(n+1)/A(n) = |T|$$

The above result can be generalized to 4 and N state problems. In a 4-state Markovian process, a tetrahedron of unit altitude from each face is constructed. The sum of the perpendiculars to the four faces from any interior point is 1.

The volume V of a simplex with N vertices embedded in N-1 dimensional space is deduced as

$$V = C\,|T|$$

By an extension of the argument for a 3-state process, the shrinkage rate of the simplex volume is again given by the absolute value of the determinant of T.

The shrinkage factor for a finite time t can be derived by repeated applications of T. Noting that the determinant of a matrix is invariant under a similarity transformation, we prove that the shrinkage factor F for time t is exp(t(Tr R)) exactly, where Tr R denotes the trace of the matrix R.

Namely, we have

$$F = \lim_{n\to\infty} |(\det T)^n|$$

$$= \lim_{n\to\infty} |\det (S\,T\,S^{-1})^n|$$

$$= \det (\exp (Rt))$$

$$= \exp (t\,(\mathrm{Tr}\ R))$$

Or more directly, noting that det T = 1+dt (Tr R) up to the first order in dt,

$$F = \lim_{n\to\infty} |(1 + dt\ (\mathrm{Tr}\ R))^n|$$

$$= \lim_{n\to\infty} |(1 + \frac{t}{n}\ (\mathrm{Tr}\ R))^n|$$

$$= \exp (t(\mathrm{Tr}\ R))$$

Once r_{ij} s are given, the trace of R is easy to compute. From Eq. (3),

$$\mathrm{Tr}\ R = - ||r||$$

where $||r|| = \sum_{ij} r_{ij}$ by definition.

NUMERICAL APPLICATION

The range of the initial estimate of p may be anywhere within a simplex of volume K containing the equilibrium solution. If the initial guess is so crude that only normalization and positiveness of elements of p are ensured, then K is the volume of the simplex with N vertices in N-1 dimensions with an altitude of 1 above every base. If the transient equation is treated as an initial value problem and solved by integrating from t = 0 with the initial estimate up to t, the volume of uncertainty in the barycentric coordinate system shrinks to K exp(- ||r||t).

When an accuracy in p is desired to the extent of an error of a simplex volume ϵ, then the integration has to be carried out from zero to an upper limit t_{max} given by

$$||r||\ t_{max} = \ln (K/\epsilon)$$

The usual truncation and roundoff errors are neglected throughout this paper. If ||r|| is small, t_{max} is big. But then big integration steps can be chosen so that the total number of steps is not too large. The better the initial estimate, the faster the convergence will be toward the equilibrium solution.

For irreducible Markov processes, it is possible to show that the shrinkage of the simplex of possible region indeed implies the convergence to the stationary solution for every component of p. However, the rate of convergence for individual components is determined predominantly by the magnitude of the smallest real part of the nonzero eigenvalue of R.

We note that the power method employed for RQA by Wallace and Rosenberg corresponds in principle to a special case of the time-dependent approach with a constant integration step size. For a nearly decomposable system (Courtois, 1975) with small eigenvalues of R, the number of iterations required is known to be very large. In fact, most of the iterative schemes investigated by Stewart (1978) were unsuccessful for nearly decomposable states except the lopsided iteration.

In the present method, it is important to choose a numerical method in which the size of the time steps varies so that bigger steps can be taken when transient states vary slowly. The integration of a system of first-order differential equations with constant coefficients is extensively discussed in the literature (Ralston, 1960). We modified a version of Hamming's predictor-corrector method from the IBM Scientific Subroutine Package (1970). In our implementation, transition matrix elements are calculated at every integration step rather than storing them in order to conserve memory space.

It is easy to prove that if p is normalized at the initial moment, it will remain normalized for all t except for the roundoff and truncation errors, even though only the homogeneous equation, Eq. (1), is integrated. This is true because of the probability conservation law and is not a general property of the system linear equations. The proof is simply obtained by summing Eq. (1) for p over i.

$$\frac{d}{dt} \sum_i p_i = 0$$

A SPECIFIC EXAMPLE

For a concrete demonstration of the techniques involved, let us consider the following example of a single closed chain with two queues. The population of the chain is M, and the service times at the queues are independent and exponentially distributed with averages of s_1 and s_2. We assume that both queues have the FCFS discipline and no more than one customer may be served at a time at each queue.

Thus, the average execution rate b_k at queue k is given by $b_k = 1/s_k$ for k = 1 and 2.

The queuing state of this system is completely characterized by i, the number of customers at queue 1, where i = 0,1,2, . . . ,M.

The expression for r_{ij} can be written, using zero origin indexing

$$(r_{ij}) = \begin{pmatrix} 0 & b_2 & 0 & 0 & . & . & . \\ b_1 & 0 & b_2 & 0 & . & . & . \\ 0 & b_1 & 0 & b_2 & . & . & . \\ 0 & 0 & b_1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & b_2 \\ . & . & . & . & . & b_1 & 0 \end{pmatrix}$$

Following the procedure leading to Eq. (2), the stationary solution for state probability vector p can be derived.

$$p_i = (b_2/b_1)^i/Z \quad \text{for } i = 0,1,\ldots,M$$

where p_i denotes the probability that there are i customers at queue 1

and

$$Z = \sum_{k=0}^{M} (b_2/b_1)^k$$

For the current example,

$$||r|| = M\ (b_1+b_2)$$

If $b_1 = 500$ and $b_2 = 25$ corresponding to $s_1 = 0.002$ sec and $s_2 = 0.04$ sec (typical for CPU and disk service times), $||r|| = 1050$ for M = 2.

Since ln (K/ϵ) is proportional to M for large values of M due to dimensional considerations, t_{max} is insensitive to M.

Assuming $K/\epsilon = 10^{12}$,

$$t_{max}=0.0263$$

Generally, convergence is rapid and integration need not be carried out very far. Note that t_{max} is the upper limit of integration that assures the specified accuracy. In practice, desired accuracy may be attained before reaching the upper limit. In the power method employed by Wallace and Rosenberg, the equilibrium state equation is formulated as an eigenvalue by writing

$$W^T p = p$$

where $W = R^T \Delta t + I$

Δt is a parameter to be chosen so that W is a stochastic matrix. This is accomplished if

$$\Delta t < 1/C$$

$$C=\max_i \sum_{j\neq i}^{N} r_{ij}$$

Then, iterative procedure based on the recursive relation

$$p^{(n+1)} = W^T p^{(n)}$$

starting from an initial approximation of $p^{(0)}$ is guaranteed to converge.

For M = 2 in the current example,

$$R=\begin{pmatrix} -b_2 & b_1 & 0 \\ b_2 & -(b_1+b_2) & b_1 \\ 0 & b_2 & -b_1 \end{pmatrix}$$

For W to be a stochastic matrix, it is necessary to choose

$$\Delta t < 1/(b_1+b_2)$$

The three eigenvalues of the eigenvalue equation

$$W^T p = w\ p$$

are found to be

$$w_1=1$$

$$w_2=1-\Delta t\ (b_1+b_2+ (b_1 b_2)^{1/2})$$

$$w_3=1-\Delta t\ (b_1+b_2- (b_1 b_2)^{1/2})$$

Choosing $\Delta t=a/(b_1+b_2)$, where $0< a <1$, we can write

$$w_2=1 - a\ \left[1+ y^{1/2}/(1+y)\right]$$

$$w_3=1 - a\ \left[1- y^{1/2}/(1+y)\right]$$ where $y = b_2/b_1$ by definition.

Since

$$0 < y^{1/2}/(1+y) < 0.5$$

we note that all eigenvalues of W which are different from one have modulus smaller than 1, in accordance with a theorem by Gershgorin (Varga, 1963). The uniqueness of the unit eigenvalue is guaranteed as long as the Markov process is irreducible.

The power method converges to the dominant eigenvector with an eigenvalue of 1. The rate of convergence is determined by the ratio of the magnitude of the subdominant eigenvalue over the dominant one. We have experimented with the convergence rate for a variety of choice of the parameters y and a. If the subdominant eigenvalue is close to 1 as a result of a choice of a small value of a and/or y, convergence is extremely slow.

Aside from the present example, nearly decomposable queues tend to have eigenvalues close to 1; therefore, the power method works poorly for this important class of problems. For our particular example, y = 0.05. The solution for vector p is specified by

$$p_0=0.9501187$$

$$p_1=0.0475059$$

$$p_2=0.0023753$$

For Hamming's integration, we tried many different initial estimates with a wide range of initial step sizes. An initial estimate may be as crude as a vector of N-1 zeros and a one. The initial step size tested ranged from 0.001 t_{max} to 0.5 t_{max}. The upper bound of the local truncation error, as defined in subroutine HPCG (IBM, 1970), was set to 0.0001. Because the program automatically halves or doubles the integration step size on the basis of the truncation error bound, the solutions are insensitive to initial step sizes. In order to obtain an accuracy up to 4 decimal digits in the current example for M=2, the integration had to be carried out to t = 0.0143 for an initial step size of $t_{max}/100$, requiring the execution of 54 variable steps. When we started with a step size of $t_{max}/20$ and integrated up to t = 0.0217, only 24 steps had to be executed.

For a sample run of Hamming's integration, refer to Table 1. The initial step size is 0.001714 corresponding to a = 0.09 in the power method. IHLF refers to the number of times the initial step size is halved.

TABLE 1 Sample Run of Hamming's Integration

t	p_0	p_1	p_3	IHLF
0.0	1.000000	0.0	0.0	0
0.000857	0.982702	0.017129	0.000169	1
0.001714	0.971609	0.027873	0.000518	1
0.002571	0.964377	0.034733	0.000889	1
0.003000	0.961762	0.037174	0.001064	2
0.003429	0.959651	0.039122	0.001227	2
0.003857	0.957937	0.040687	0.001376	2
0.004286	0.956541	0.041949	0.001510	2
0.004714	0.955402	0.042969	0.001629	2
0.005143	0.954471	0.043796	0.001733	2
0.005571	0.953708	0.044468	0.001825	2
0.006000	0.953082	0.045014	0.001904	2
0.006857	0.952154	0.045811	0.002035	1
0.007714	0.951517	0.046354	0.002129	1
0.008571	0.951081	0.046720	0.002198	1
0.009429	0.950785	0.046966	0.002249	1
0.010286	0.950578	0.047137	0.002285	1
0.011143	0.950437	0.047252	0.002311	1
0.012000	0.950341	0.047329	0.002330	1
0.012857	0.950273	0.047384	0.002343	1
0.013714	0.950227	0.047420	0.002353	1
0.014572	0.950195	0.047446	0.002359	1
0.016286	0.950150	0.047484	0.002366	0
0.018000	0.950136	0.047494	0.002370	0
0.019714	0.950135	0.047490	0.002374	0
0.021429	0.950123	0.047503	0.002374	0
0.023143	0.950118	0.047507	0.002374	0
0.024857	0.950129	0.047494	0.002377	0
0.026572	0.950118	0.047507	0.002375	0

With the power method, we needed only 4 iterations with the choice of a = 0.99 to achieve the same accuracy as before. For a = 0.09, however, 109 iterations had to be performed, reflecting the fact that the subdominant eigenvalues of W are close to 1 (i.e., $w_1 = 0.904$ and $w_2 = 0.992$). Furthermore, the convergent eigenvector had to be renormalized.

We have tested up to M = 5000. Generally, we found Hamming's integration to be more effective than the power method if the subdominant eigenvalues are close to 1, but the power method performed better in other cases. In the range of the parameters tested, we saw no evidence of numerical instability.

CONCLUSION

The geometrical interpretation used in this work is helpful in understanding the convergence rate of transient Markovian states toward equilibrium states. The results derived from this paper may be used for estimating the stopping time for simulation of Markov processes. It would be interesting to apply the result of this work to the question of how much time is involved in a real computer system for the workload and system behavior to reach a steady state commonly assumed in performance model validations.

The numerical approach taken in this paper offers an alternative to the existing methods, even for certain types of analytical queuing problems which may not be soluble with any known special techniques such as the product form solutions. However, in common with all direct numerical solution techniques for queuing networks, much progress needs to be made to overcome space and computation time problems for large states.

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